Fork-Join Pattern

Parallel Computing
CIS 410/510
Department of Computer and Information Science
Outline

- What is the fork-join concept?
- What is the fork-join pattern?
- Programming Model Support for Fork-Join
- Recursive Implementation of Map
- Choosing Base Cases
- Load Balancing
- Cache Locality and Cache-Oblivious Algorithms
- Implementing Scan with Fork-Join
- Applying Fork-Join to Recurrences
Fork-Join Philosophy

When you come to a fork in the road, take it.

(Yogi Bera, 1925 –)
Fork-Join Concept

- Fork-Join is a fundamental way (primitive) of expressing concurrency within a computation
- **Fork** is called by a (logical) thread (*parent*) to create a new (logical) thread (*child*) of concurrency
  - Parent continues after the *Fork* operation
  - Child begins operation separate from the parent
  - *Fork* creates concurrency
- **Join** is called by both the parent and child
  - Child calls *Join* after it finishes (implicitly on exit)
  - Parent waits until child joins (continues afterwards)
  - *Join* removes concurrency because child exits
Fork-Join Concurrency Semantics

- Fork-Join is a concurrency control mechanism
  - Fork increases concurrency
  - Join decreases concurrency

- Fork-Join dependency rules
  - A parent must join with its forked children
  - Forked children with the same parent can join with the parent in any order
  - A child cannot join with its parent until it has joined with all of its children

- Fork-Join creates a special type of DAG
  - What do they look like?
Fork Operation

- Fork creates a child thread
- What does the child do?
- Typically, fork operates by assigning the child thread with some piece of “work”
  - Child thread performs the piece of work and then exits by calling join with the parent
- Child work is usually specified by providing the child with a function to call on startup
- Nature of the child work relative to the parent is not specified
Join Operation

- Join informs the parent that the child has finished
- Child thread notifies the parent and then exits
  - Might provide some status back to the parent
- Parent thread waits for the child thread to join
  - Continues after the child thread joins
- Two scenarios
  1. Child joins first, then parent joins with no waiting
  2. Parent joins first and waits, child joins and parent then continues
Fork-Join Heritage in Unix

- Fork-Join comes from basic forms of creating processes and threads in operating system
- Forking a child process from a parent process
  - Creates a new child process with `fork()`
  - Process state of parent is copied to child process
    - process ID of parent stored in child process state
    - process ID of child stored in parent process state
  - Parent process continues to next PC on `fork()` return
  - Child process starts execution at next PC
    - process ID is automatically set to child process
    - child can call `exec()` to overlay another program
Fork-Join Heritage in Unix (2)

- Joining a child process with a parent process
  - Child process exits and parent process is notified
    - if parent is blocked waiting, it unblocks
    - if parent is not waiting, some indication is made
    - child process effectively joins
  - Parent process calls waitpid() (effectively join) for a particular child process
    - if the child process has called join(), parent continues
    - if the child process has not called join(), parent blocks
- Fork-Join also implemented for threads
Fork-Join “Hello World” in Unix

```c
#include <sys/types.h> /* pid_t */
#include <sys/wait.h> /* waitpid */
#include <stdio.h> /* printf, perror */
#include <stdlib.h> /* exit */
#include <unistd.h> /* _exit, fork */

int main(void)
{
    pid_t pid;

    pid = fork();

    if (pid == -1) {
        /*
         * When fork() returns -1, an error happened.
         */
        perror("fork failed");
        exit(EXIT_FAILURE);
    } else if (pid == 0) {
        /*
         * When fork() returns 0, we are in the child process.
         */
        printf("Hello from the child process!\n");
        _exit(EXIT_SUCCESS); /* exit() is unreliable here, so _exit must be used */
    } else {
        /*
         * When fork() returns a positive number, we are in the parent process
         * and the return value is the PID of the newly created child process.
         */
        int status;
        (void)waitpid(pid, &status, 0);
    }
    return EXIT_SUCCESS;
}
```
Fork-Join in POSIX Thread Programming

- POSIX standard multi-threading interface
  - For general multi-threaded concurrent programming
  - Largely independent across implementations
  - Broadly supported on different platforms
  - Common target for library and language implementation

- Provides primitives for
  - Thread creation and management
  - Synchronization
### Thread Creation

```c
#include <pthread.h>
int pthread_create(
    pthread_t *thread_id,
    const pthread_attr_t *attribute,
    void *(*thread_function)(void *),
    void *arg);
```

- **thread_id**
  - thread’s unique identifier
- **attribute**
  - contain details on scheduling policy, priority, stack, ...
- **thread_function**
  - function to be run in parallel (entry point)
- **arg**
  - arguments for function `func`
Example of Thread Creation

```c
void *func(void *arg) {
    int *I=arg;
    ...
}

void main() {
    int X;
    pthread_t id;
    ...
    pthread_create(&id, NULL, func, &X);
    ...
}
```
Pthread Termination

```c
void pthread_exit(void *status)
```

- Terminates the currently running thread
- Implicitly called when function called in `pthread_create` returns
Thread Joining

```c
int pthread_join(
    pthread_t thread_id,
    void **status);
```

- Waits for thread `thread_id` to terminate
  - Either by returning
  - Or by calling `pthread_exit()`

- Status receives the return value or the value given as argument to `pthread_exit()`
Thread Joining Example

```c
void *func(void *){
    ... 
}

pthread_t id;
int X;
...

pthread_create(&id, NULL, func, &X);
...

pthread_join(id, NULL);
... 
```
General Program Structure

- Encapsulate parallel parts in functions
- Use function arguments to parameterize thread behavior
- Call `pthread_create()` with the function
- Call `pthread_join()` for each thread created
- Need to take care to make program “thread safe”
Pthread Process Management

- `pthread_create()`
  - Creates a parallel thread executing a given function
  - Passes function arguments
  - Returns thread identifier

- `pthread_exit()`
  - Terminates thread.

- `pthread_join()`
  - Waits for particular thread to terminate
Pthreads Synchronization

- Create/exit/join
  - Provide some coarse form of synchronization
  - “Fork-join” parallelism
  - Requires thread creation/destruction

- Need for finer-grain synchronization
  - Mutex locks
  - Condition variables
```c
#include <pthread.h>
#include <stdio.h>
#include <stdlib.h>
#include <assert.h>

#define NUM_THREADS 5

void *TaskCode(void *argument)
{
    int tid;
    tid = *(int *) argument;
    printf("Hello World! It's me, thread %d\n", tid);
    /* optionally: insert more useful stuff here */
    return NULL;
}

int main(void)
{
    pthread_t threads[NUM_THREADS];
    int thread_args[NUM_THREADS];
    int rc, i;

    /* create all threads */
    for (i=0; i<NUM_THREADS; ++i) {
        thread_args[i] = i;
        printf("In main: creating thread %d\n", i);
        rc = pthread_create(&threads[i], NULL, TaskCode, (void *)&thread_args[i]);
        assert(0 == rc);
    }

    /* wait for all threads to complete */
    for (i=0; i<NUM_THREADS; ++i) {
        rc = pthread_join(threads[i], NULL);
        assert(0 == rc);
    }

    exit(EXIT_SUCCESS);
}
```
**Fork-Join Pattern**

- Control flow divides (forks) into multiple flows, then combines (joins) later

- During a fork, one flow of control becomes two

- Separate flows are “independent”
  - Does “independent” mean “not dependent”?
  - No, it just means that the 2 flows of control “are not constrained to do similar computation”

- During a join, two flows become one, and only this one flow continues
Fork-Join Pattern

- Fork-Join directed graph:

  - Independent work

  Is it possible for B() and C() to have dependencies between them?
Fork-Join Pattern

- Typical **divide-and-conquer** algorithm implemented with fork-join:

```c
void DivideAndConquer( Problem P ) {
  if( P is base case ) {
    Solve P;
  } else {
    Divide P into K subproblems;
    Fork to conquer each subproblem in parallel;
    Join;
    Combine subsolutions into final solution;
  }
}
```
Fork-Join Pattern for Divide-Conquer
Fork-Join Pattern for Divide-Conquer

\[ K = 2 \quad \text{(2-way fork-join)} \]

\[ N = 3 \quad \text{(3 levels of fork-join)} \]
Fork-Join Pattern for Divide-Conquer

2 \times 3 = 8 \text{-way parallelism}
Fork-Join Pattern

- Selecting the base case size is critical
- Recursion must go deep enough for plenty of parallelism
- Too deep, and the granularity of sub-problems will be dominated by scheduling overhead
- With K-way fork-join and N levels of fork-join, can have up to $K^N$-way parallelism
Fibonacci Example

- Recursive Fibonacci is simple and inefficient

```c
long fib ( int n ) {
    if (n < 2) return 1;
    else {
        long x = fib (n-1);
        long y = fib(n-2);
        return x + y;
    }
}
```
**Fibonacci Example**

- Recursive Fibonacci is simple and inefficient
- Are there dependencies between the sub-calls?
- Can we parallelize it?
Fibonacci in Parallel Example

long fib ( int n ) {
    if (n < 2) return 1;
else {
    long x = fork fib (n-1);
    long y = fib(n-2);
    join;
    return x + y;
}
}
Programming Model Support for Fork-Join

- Cilk Plus:

  ```cilk
  cilk_spawn B();  // Fork
  C();
  cilk_sync;  // Join
  ```

- B() executes in the child thread
- C() executes in the parent thread
Programming Model Support for Fork-Join

- Cilk Plus:

```c
  cilk_spawn B();
  C();
  cilk_sync;
  cilk_spawn A();
  cilk_spawn B();
  cilk_spawn C();
  D();      // Not spawned, executed in spawning task
  cilk_sync;  // Join
  for ( int i=0; i<n; ++i )
    if ( a[i]!=0 )
      cilk_spawn f(a[i]);
  cilk_sync;
```

Good form!
Programming Model Support for Fork-Join

- Cilk Plus:

```c
    cilk_spawn B();
    cilk_spawn C();
    /* nil */
    cilk_sync;
```

Bad form! Why?

Diagram:

```
    B()  C()  Nil
    ↓    ↓    ↓
    ↓    ↓    ↓
```

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Programming Model Support for Fork-Join

- **TBB**
  - `parallel_invoke()`
    - For 2 to 10 way fork
    - Joins all tasks before returning
  - `Tbb::task_group`
    - For more complicated cases
    - Provides explicit join

```cpp
    task_group g;
    for ( int i=0; i<n; ++i )
        if ( a[i] != 0 )
            g.run( [=,&a]{f(a[i]);} );  // Spawn f(a[i]) as child task
    g.wait();                      // Wait for all tasks spawned from g
```
Programming Model Support for Fork-Join

- OpenMP:

  ```c
  #pragma omp task
  B();
  C();
  #pragma omp taskwait
  ```

  Forked task

  Performed by spawning task
Programming Model Support for Fork-Join

- OpenMP:

  ```c
  #pragma omp task
  B();
  C();
  #pragma omp taskwait
  ```

  Forked task can also be a compound statement:
  ```c
  { B(); C(); D(); }
  ```
Programming Model Support for Fork-Join

- OpenMP:

```c
#pragma omp task
B();
C();
#pragma omp taskwait
```

Must be enclosed in an OpenMP parallel construct
More to the OpenMP Fork-Join Story

- OpenMP uses a fork-join model of parallel execution as a fundamental basis of the language
- All OpenMP programs begin as a single process
  - *Master* thread executes until a parallel region is encountered
- OpenMP runtime systems executes the parallel region by forking a team of (*Worker*) parallel threads
  - Statements in parallel region are executed by worker threads
- Team threads join with master at parallel region end
OpenMP – General Rules

- Most OpenMP constructs are *compiler directives*

- Directives **inform** the compiler
  - Provide compiler with knowledge
  - Usage assumptions

- Directives are ignored by non-OpenMP compilers!
  - Essentially act as comment for backward compatibility

- Most OpenMP constructs apply to structured blocks
  - A block of code with one point of entry at the top and one point of exit at the bottom
  - Loops are a common example of structured blocks
    - excellent source of parallelism
OpenMP PARALLEL Directive

- Specifies what should be executed in parallel:
  - A program section (structured block)
  - If applied to a loop, what happens is:
    - iterations are executed in parallel
    - `do` loop (Fortran)
    - `for` loop (C/C++)

- PARALLEL DO is a “worksharing” directive
  - Causes work to be shared across threads
  - More on this later
PARALLEL DO: Syntax

- **Fortran**
  
  ```fortran
  !$omp parallel do [clause [,] [clause ...]]
  do index = first, last [, stride]
    body of the loop
  enddo
  [!$omp end parallel do]
  ```
  The loop body executes in parallel across OpenMP threads

- **C/C++**
  
  ```c++
  #pragma omp parallel for [clause [clause ...]]
  for (index = first; text_expr;
    increment_expr) {
    body of the loop
  }
  ```
Example: PARALLEL DO

- Single precision $a \times x + y$ (saxpy)

```fortran
subroutine saxpy (z, a, x, y, n)
integer i, n
real z(n), a, x(n), y(n)
!$omp parallel do
do i = 1, n
    z(i) = a * x(i) + y(i)
enddo
return
end
```

What is the degree of concurrency?

What is the degree of parallelism?
**Execution Model of PARALLEL DO**

- Master thread executes serial portion of code
- Master thread enters *saxpy* routine
- Master thread encounters *parallel do* directive
- Creates slave threads (How many?)
- Master and slave threads divide iterations of parallel do loop and execute them concurrently

Implicit synchronization: wait for all threads to finish their allocation of iterations
- Master thread resumes execution after the do loop
- Slave threads disappear

- Abstract execution model – a Fork-Join model!!!
Loop-level Parallelization Paradigm

- Execute each loop in parallel
  - Where possible
- Easy to parallelize code
- Incremental parallelization
  - One loop at a time
  - What happens between loops?
- Fine-grain overhead
  - Frequent synchronization
- Performance determined by sequential part (Why?)

```c
C$OMP PARALLEL DO
do i=1,n

alpha = xnorm/sum
C$OMP PARALLEL DO
do i=1,n

C$OMP PARALLEL DO
do i=1,n
```

```c
enddo
........
enddo
```
Example: PARALLEL DO – Bad saxpy

- Single precision \( a \times x + y \) (saxpy)

```fortran
subroutine saxpy (z, a, x, y, n)
    integer i, n
    real z(n), a, x(n), y(n)

    !$omp parallel do
    do i = 1, n
        y(i) = a * x(i+1) + y(i+1)
    enddo
    return
end
```

What happens here?
How Many Threads?

- Use environment variable
  - `setenv OMP_NUM_THREADS 8` (Unix machines)
- Use `omp_set_num_threads()` function

```fortran
subroutine saxpy (z, a, x, y, n)
  integer i, n
  real z(n), a, x(n), y(n)

!$omp parallel do
call omp_set_num_threads(4)
%!omp parallel do
do i = 1, n
  z(i) = a * x(i) + y(i)
enddo
return
end
```

Not a directive, but a call to the OpenMP library
Assigning Iterations to Threads

- A parallel loop in OpenMP is a worksharing directive
- The manner in which iterations of a parallel loop are assigned to threads is called the loop’s schedule
- Default schedule assigns iterations to threads as evenly as possible (good enough for saxpy)
- Alternative user-specified schedules possible
- More on scheduling later
PARALLEL DO: The Small Print

- The programmer has to make sure that the iterations can in fact be executed in parallel
  - No automatic verification by the compiler

```fortran
subroutine noparallel (z, a, x, y, n)
  integer i, n
  real z(n), a, x(n), y(n)
!$omp parallel do
  do i = 2, n
    z(i) = a * x(i) + y(i) + z(i-1)
  enddo
return
end
```

---

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**PARALLEL Directive**

- **Fortran**
  ```fortran
  !$omp parallel [clause [,] [clause ...]]
  structured block
  !$omp end parallel
  ```

- **C/C++**
  ```c
  #pragma omp parallel [clause [clause ...]]
  structured block
  ```
Parallel Directive: Details

- When a parallel directive is encountered, threads are spawned which execute the code of the enclosed structured block (i.e., the parallel region)
- The number of threads can be specified just like for the PARALLEL DO directive
- The parallel region is replicated and each thread executes a copy of the replicated region
Example: Parallel Region

```c
double A[1000];
omp_set_num_threads(4);
#pragma omp parallel
{
    int ID = omp_thread_num();
    pooh(ID, A);
}
printf("all done\n");
```

```c
ID = omp_thread_num() ... ID = omp_thread_num()
pooh(0,A) pooh(1,A) pooh(2,A) pooh(3,A)
```

Is this ok?

```c
printf("all done\n");
```
Parallel versus Parallel Do

- Arbitrary structured blocks versus loops
- Coarse grained versus fine grained
- Replication versus work division (work sharing)

```
!$omp parallel do
do I = 1,10
    print *, 'Hello world', I
enddo

!$omp parallel do
output: 10 Hello world messages

!$omp parallel
!$omp do
output: 10*T Hello world messages
where T = number of threads
```

PARALLEL DO is a work sharing directive
Parallel: Back to Motivation

omp_set_num_threads(2);
#pragma omp parallel private(i, j, x, y, my_width, my_thread, i_start, i_end)
{
    my_width = m/2;
    my_thread = omp_get_thread_num();
    i_start = 1 + my_thread * my_width;
    i_end = i_start + my_width - 1;
    for (i = i_start; i <= i_end; i++)
        for (j = 1; j <= n; j++) {
            x = i/ (double) m;
            y = j/ (double) n;
            depth[j][i] = mandel_val(x, y, maxiter);
        }
    for (i = i_start; i <= i_end; i++)
        for (j = 1; j <= n; j++)
            dith_depth[j][i] = 0.5*depth[j][i]
                             + 0.25*(depth[j-1][i] + depth[j+1][i])
}

What is going on here?
Work Sharing in Parallel Regions

- Manual division of work (previous example)
- OMP worksharing constructs
  - Simplify the programmers job in dividing work among the threads that execute a parallel region
    - **do** directive
      - have different threads perform different iterations of a loop
    - **sections** directive
      - identify sections of work to be assigned to different threads
    - **single** directive
      - specify that a section of code is to be executed by one thread only
        (remember default is replicated)
**DO Directive**

- **Fortran**
  ```fortran
  !$omp parallel [clause [,] [clause ...]]
  ...
  !$omp do [clause [,] [clause ...]]
    do loop
  !$omp enddo [nowait]
  ...
  !$omp end parallel
  ```

- **C/C++**
  ```c
  #pragma omp parallel [clause [clause ...]]
  {
    ...
    #pragma omp for [clause [clause] ... ]
      for-loop
  }
  ```
DO Directive: Details

- The DO directive does not spawn new threads!
  - It just assigns work to the threads already spawned by the PARALLEL directive
- The work↔thread assignment is identical to that in the PARALLEL DO directive

```c
!$omp parallel do
do I = 1,10
  print *, 'Hello world', I
endo do

!$omp parallel do
do I = 1,10
  print *, 'Hello world', I
endo do

!$omp end parallel
```
Coarser-Grain Parallelism

What’s going on here? Is this possible? When?

Is this better? Why?
**SECTIONS Directive**

- **Fortran**
  ```fortran
  !$omp sections [clause [,] [clause ...]]
  !$omp section
  code for section 1
  !$omp section
  code for section 2
  ...
  !$omp end sections [nowait]
  ```

- **C/C++**
  ```c
  #pragma omp sections [clause [clause ...]]
  {
    [#pragma omp section]
    block
    ...
  }
  ```
SECTIONS Directive: Details

- Sections are assigned to threads
  - Each section executes once
  - Each thread executes zero or more sections
- Sections are not guaranteed to execute in any order

```c
#pragma omp parallel
#pragma omp sections
{
    X_calculation();
    #pragma omp section
    y_calculation();
    #pragma omp section
    z_calculation();
}
```
OpenMP Fork-Join Summary

- OpenMP parallelism is Fork-Join parallelism
- Parallel regions have logical Fork-Join semantics
  - OMP runtime implements a Fork-Join execution model
  - Parallel regions can be nested!!!
    - can create arbitrary Fork-Join structures
- OpenMP tasks are an explicit Fork-Join construct
Recursive Implementation of Map

- **Map** is a simple, useful pattern that fork-join can implement
- Good to know how to implement map with fork-join if you ever need to write your own map with novel features (fusing map with other patterns)
- Cilk Plus and TBB implement their map constructs with a similar divide-and-conquer algorithm
Recursive Implementation of Map

cilk_for( unsigned i=lower; i<upper; ++i )
f(i);

cilk_for can be implemented with a divide-and-conquer routine…

if( lower<upper )
    recursive_map(lower,upper,grainsize,f)
Recursive Implementation of Map

1 template<typename Func>
2 void recursive_map( unsigned lower, unsigned upper, unsigned grainsize, Func f ) {
3     if( upper-lower <= grainsize )
4         // Parallel base case
5             for( unsigned i=lower; i<upper; ++i )
6                 f(i);
7     else {
8         // Divide and conquer
9             unsigned middle = lower+(upper-lower)/2u;
10             cilk_spawn recursive_map( lower, middle, grainsize, f );
11             recursive_map( middle, upper, grainsize, f );
12         }
13         // Implicit cilk_sync when function returns
14     }

Recursive Implementation of Map

```python
recursive_map(0, 9, 2, f)
```
Choosing Base Cases (1)

- For parallel divide-and-conquer, two base cases:
  - Stopping parallel recursion
  - Stopping serial recursion
- For a machine with \( P \) hardware threads, we might think to have \( P \) leaves in the spawned functions tree
- This often leads to poor performance
  - Scheduler has no flexibility to balance load
Choosing Base Cases (2)

- Given leaves from spawned function tree with equal work, and equivalent processors, system effects can effect load balance:
  - Page faults
  - Cache misses
  - Interrupts
  - I/O

- Best to over-decompose a problem
- This creates parallel slack
Choosing Base Cases (3)

- **Over-decompose**: parallel programming style where more tasks are specified than there are physical workers. Beneficial in load balancing.

- **Parallel slack**: Amount of extra parallelism available above the minimum necessary to use the parallel hardware resources.
Load Balancing

- Sometimes, threads will finish their work at different rates.
- When this happens, some threads may have nothing to do while others may have a lot of work to do.
- This is known as a load balancing issue.
**TBB and Cilk Plus Work Stealing (1)**

- TBB and Cilk Plus use *work stealing* to automatically balance fork-join work
- In a work-stealing scheduler, each thread is a *worker*
- Each worker maintains a stack of tasks
- When a worker’s stack is empty, it grabs from the *bottom* of another random worker
  - Tasks at the bottom of a stack are from the beginning of the call tree – tend to be a bigger piece of work
  - Stolen work will be distant from stack’s owner, minimizing cache conflicts
TBB and Cilk Plus Work Stealing (2)

- TBB and Cilk Plus work-stealing differences:

Cilk Plus

TBB

Steal continuation

Steal child
Performance of Fork/Join

Let $A \parallel B$ be interpreted as “fork $A$, do $B$, and join”

Work: $T(A \parallel B)_1 = T(A)_1 + T(B)_1$

Span: $T(A \parallel B)_\infty = \max(T(A)_\infty, T(B)_\infty)$
Cache-Oblivious Algorithms (1)

- Work/Span analysis ignores memory bandwidth constraints that often limit speedup.
- Cache reuse is important when memory bandwidth is critical resource.
- Tailoring algorithms to optimize cache reuse is difficult to achieve across machines.

- **Cache-oblivious programming** is a solution for this.
- Code is written to work well regardless of cache structure.
Cache-Oblivious Algorithms (2)

- Cache-oblivious programming strategy:
  - Recursive divide-and-conquer – good data locality at multiple scales
  - When a problem is subdivided enough, it can fit into the largest cache level
  - Continue subdividing to fit data into smaller and faster cache

- Example problem: matrix multiplication
  - Typical, non-recursive, algorithm uses three nested loops
  - Large matrices won’t fit in cache with this approach
Implementing Scan with Fork-Join

- We saw that the map pattern can be implemented with the fork-join pattern
- Now we will examine how to implement the scan operation with fork-join
- Input: initial value, initial, and sequence, \( r↓0 \), \( r↓1 \), ..., \( r↓n \)
- Output: exclusive scan, \( s↓0 \), \( s↓1 \), ..., \( s↓n \)
- Upsweep computes a set of partial reductions on tiles of data
- Downsweep computes final scan by combining partial reductions
Implementing Scan with Fork-Join
Implementing Scan with Fork-Join
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- During the upsweep, each node computes a partial reduction of the form:
  \[ r_{i:m} = r_{i:k} \oplus r_{i+k:m-k} \]
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- During the downsweep, each node computes subscans of the form:
  \[ s_{\downarrow i} = \text{initial} \oplus r_{\downarrow 0:i} \quad \text{and} \quad s_{\downarrow i+k} = s_{\downarrow i} \oplus r_{\downarrow i:k} \]